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Results Concerning Exact Controllability of Nonlinear Volterra-Fredholm Integro-differential Equations of Fractional Order Via a Semigroup Approach

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Abstract

This manuscript explored the exact controllability of nonlinear nonlocal Volterra–Fredholm fractional integro-differential equations (NNVFFIE) in Banach spaces. Inspired the effectiveness of fractional-order models to capture memory and hereditary effects in biological, engineering, and physical systems, a fractional control system with the Caputo derivative is considered due to its relevance for initial value problems. Using tools from fractional calculus and the theory of strongly continuous semigroups, the control system is reformulated in an abstract integral form. Sufficient conditions for exact controllability are established using a fixed point theorem under appropriate compactness, boundedness, and growth assumptions on the nonlinear terms. The proposed framework provides a systematic approach to steering the system from a given initial state to a desired final state within a finite time interval. The obtained results extend existing exact controllability criteria for fractional integro-differential systems. An illustrative example is included to demonstrate the applicability of the abstract findings.

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Keywords

NNVFFIE, Exact controllability, Fractional calculus, Fixed point theorem, Semigroup theory

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1. Introduction

Recently fractional differential equations (FDEs) have emerged as an essential tool for modeling complex dynamical processes which demonstrate long-term memory, nonlocality, and hereditary effects. In this context, FDEs constitute an important extension of classical models based on integer-order calculus. These equations contain derivatives and integrals of non-integer order. Because of this, such equations effectively capture the temporal correlations inherent in several physical, engineering and biological processes. Applications include diffusion processes, heat conduction, electrochemistry, control systems and signal processing governed by complex dynamics [1-5]. In recent decades, significant research has been done in FDEs by many researchers; see the monograph of [6,7] and the papers [8-11].

Fractional integro-differential equations (FIEs) have attracted increasing attention because of their ability to incorporate both integral memory terms and differential operators within a unified framework. Such equations are significantly important for systems whose future states depend not only on their current states but also on distributed histories of past events. These equations generally arise in models of thermal processes with after effects, biological systems exhibiting anomalous diffusion, population dynamics and viscoelastic materials [12-15].

Moreover, in the realm of control theory, the notion of controllability plays an indispensable role in determining whether a given system can be driven from an initial state to a specified final state through admissible control inputs. Establishing controllability for fractional systems is more difficult than in the classical case. This is because of the presence of nonlocal operators and memory terms that generally make direct analytical treatment very challenging.

There are various mathematical tools for the investigation of such fractional control systems, which have been developed to handle these complexities. Semigroup theory is one of the powerful mathematical frameworks for studying such systems. When the infinitesimal generator of a strongly continuous semigroup (SCS) is associated with the governing operator, the solution of fractional differential systems can be expressed into an equivalent integral form which is suitable for theoretical study. Combined with fixed point theorems and fractional calculus, this approach has been widely used to derive controllability results for fractional integro-differential systems.

Building on these theoretical tools, many researchers [16-20] have derived sufficient conditions for the controllability of fractional integro-differential systems (FISs) in Banach spaces. Hussain et al. [21] focused on analyzing fractional-order evolution equations in Banach spaces under nonlocal constraints. Their principal aim was to study the existence of positive mild solutions and to explore the controllability of the system. The proofs rely on Schauder's and Krasnoselskii's fixed point principles, together with the Arzelà–Ascoli compactness criterion. In [22], the authors explored the controllability of impulsive fractional systems incorporating infinite state-dependent delays within an abstract Banach space setting. Their results were developed using semigroup theory in combination with Schaefer's fixed point theorem.

In [23], the controllability of FIEs incorporating state-dependent delays is analyzed, and sufficient conditions are obtained within the framework of fractional calculus and Sadovskii's fixed point theorem. The controllability of linear and nonlinear FISs is explored in [24]. The analysis starts with the linear system, where the controllability framework is developed, and the results are subsequently extended to the nonlinear system employing Schauder's fixed point theorem. Cheng et al. [25] addressed the exact controllability of fractional evolution equations with time-varying delays. By applying nonlocal conditions, the authors demonstrate the system's exact controllability through the Leray–Schauder alternative theorem and the framework of propagation families in a Banach space.

Inspired by the above work and the work cited in [26-28], the present study focuses on the exact controllability analysis of NNVFFIE within the framework of Banach spaces. The system under consideration involves a Caputo fractional derivative, which is especially appropriate for modeling initial value problems since the derivative of a constant vanishes. This investigation employs semigroup theory and the Leray–Schauder nonlinear alternative to derive sufficient conditions ensuring the system's exact controllability under appropriate assumptions. Moreover, the use of fractional integral operators allows the transformation of the original differential equation into an equivalent integral formulation, which is then examined to guarantee the existence of control functions capable of steering the system to a desired final state.

2. Problem Statement

Motivated by the work discussed above, we consider an NIIETVD of the following form:

$$\frac{d^{\gamma}}{dt^{\gamma}}y(t)=Ay(t)+J\left(t,y(t),\int_{0}^{t}m_{1}(t,\theta,y(\theta))d\theta,\int_{0}^{c}m_{2}(t,\theta,y(\theta))d\theta\right)+\Upsilon x(t),t\in D=[0,c] \quad (1)$$

$$y(0)+\varsigma(y)=y_{0}, \quad (2)$$

where $m_1, m_2 : \Delta \times V \to V$, $\Delta = \{(t,\theta) | 0 \le \theta \le t \le c\}$, $J : D \times V \times V \times V \to V$ are the functions defined in Equation (1). Also, $\varsigma : C[D,V] \to V, y_0 \in V$ and $0 < \gamma < 1$. Let A denote the infinitesimal generator of a strongly continuous semigroup $N(t), t \ge 0$. The state function $y(.)$ takes values in a real Banach space V . Moreover, the control function $x(.)$ belongs to the space $L^2(D,B)$, which is a Banach space consisting of admissible control functions. Here B is itself a Banach space. Finally, Υ represents a bounded linear operator mapping from B to V .

This study focuses on exploring the NVFFIE problem specified in Equations (1)-(2) within a Banach space setting. In addition to conventional techniques, the analysis employs modern mathematical frameworks–including the idea of semigroup theory, fractional calculus, and fixed point methods.

The present work extends existing controllability results by establishing exact controllability to a class of NNVFFIE in Banach spaces. Unlike prior studies which often impose strong Lipschitz or compactness conditions, our analysis allows more general nonlinearities satisfying Carathéodory-type conditions and weaker boundedness requirements. Moreover, the combined presence of both Volterra and Fredholm integral terms together with a Caputo fractional derivative is handled within a unified semigroup framework. This approach relaxes several structural assumptions commonly adopted in previous works and extends the applicability of controllability theory to systems with stronger memory effects and nonlocal interactions.

The principal contribution of this work lies in demonstrating how semigroup theory and fractional operators can be used to explore the exact controllability in an abstract setting. The results given here provide the theoretical basis of fractional control systems and present analytical tools which may be adapted to problems involving nonlocal constraints and memory effects.

The remainder of the work is ordered as follows. The essential mathematical preliminaries, including basic definitions and properties of semigroup theory, fractional calculus and fixed point theorems are introduced in Section 2. Section 3 provides the key exact controllability results along with the sufficient conditions that are vital for the existence of control functions. A practical example that validates the theoretical results and highlights the relevance of the approach is discussed in Section 4. Finally, the implications and potential extensions of the work are emphasized in concluding remarks.

3. Preamble and Principle Concepts

This section introduces the basic concepts and results from fixed point theory, fractional calculus, and semigroup theory that are essential for the analysis. The definitions of infinitesimal generators, strongly continuous semigroups, Caputo derivatives, and fractional integrals are recalled for completeness.

Definition 1 ([29]): Consider $N(t)$, $0 \le \varpi < \infty$ be a one parameter family of bounded linear operators (BLOs) from V into V . This family is known as semigroup of BLOs on V if it holds the following conditions:

(1) $N(0) = I$, where I represents the identity operator in V ,

(2) $N(t)N(\theta) = N(t+\theta)$, $\forall t, \theta \ge 0$.

The semigroup. $N(t)$ is called uniformly continuous if

$$\lim_{t\to 0} \|N(t) - I\| = 0 .$$

The operator A which is linear and is given by

$$\text{Domain}(A) = \left\{ y \in V \middle| \lim_{i\to 0} \frac{N(i)(y) - y}{i} \text{ exists} \right\}$$

and

$$A(y) = \lim_{i\to 0} \frac{N(i)(y) - y}{i} \text{, for } y \in dom(A) .$$

Here $A(y)$ is referred to as the infinitesimal generator of the semigroup $N(t)$.

Definition 2 ([29]): A family of BLOs $N(t), t \in [0,\infty)$ on a Banach space V is called a strongly continuous semigroup (SCS)or C_0 if it satisfies

$$\lim_{t\to 0} N(t) y = y, \forall y \in V .$$

A SCS of BLOs on V is referred to as a C_0 semigroup.

Theorem 1([29]): Consider $N(t)$ is a C_0 semigroup. Then $\exists$ a constants $L^* \geq 1$ and $\alpha \geq 0$ in such a way that

$\|N(t)\| \leq L^* e^{\alpha t}, \forall t \geq 0$. When $\alpha = 0$, the semigroup $N(t)$ is said to be uniformly bounded. Further, if $L^* = 1$, it is known as a C_0 semigroup of contractions.

Moreover, the following auxiliary lemma is required.

Lemma 1: Consider $N(t)$ be a C_0 semigroup for $t \geq 0$ on a Banach space V and be compact for $t > 0$ (Therefore $N(t)$ is continuous in the uniform operator topology for $t > 0$). Then, for every $s \geq 0$ and $y \in V$, we have

$\lim_{t \to 0} \|N(s+t)y - N(s)y\| = 0$.

Proof. Let $s, t \geq 0$ and $y \in V$. Since $(N(t))_{t \geq 0}$ is a strongly continuous semigroup, by definition, we have

$\lim_{t \to 0} N(t)z = z$ for all $z \in V$.

Applying the semigroup property, we write

$N(s+t)y = N(s)N(t)y$.

Hence,

$\|N(s+t)y - N(s)y\| = \|N(s)N(t)y - N(s)y\| = \|N(s)(N(t)y - y)\|$.

Since $N(s)$ is a bounded linear operator, $\exists\ \iota > 0$ such that

$\|N(s)z\| \leq \iota \|z\|$ for all $z \in V$.

Therefore,

$\|N(s+t)y - N(s)y\| \leq \iota \|N(t)y - y\|$.

By the strong continuity of the semigroup at $t = 0$, we have

$\lim_{t \to 0} \|N(t)y - y\| = 0$.

Hence,

$\lim_{t \to 0} \|N(s+t)y - N(s)y\| = 0$.

Furthermore, we will state some definition from fractional calculus.

Definition 3 ([30]): A real function $g(t)$ is defined in the space $C_\gamma, \gamma \in \mathbb{R}$ if $\exists$ a real number $q > \gamma$ in such a way that $g(t) = t^q f(t)$, where $f \in C[0,\infty]$ and when $g^{(p)} \in C_\gamma, p \in \mathbb{N}$, then it is called in the space C_γ^p .

Definition 4 ([31]): For $\iota > 0$, the Riemann-Liouville (RL) fractional integral operator acting on a function $g \in C_\gamma, \gamma \geq -1$ is expressed as follows

$$I^\iota g(t) = \frac{1}{\Gamma(\iota)} \int_0^t (t-\theta)^{\iota - 1} g(\theta) d\theta \text{ , where } \Gamma(\iota) = \int_0^\infty t^{\iota - 1} \exp(-t) dt, \iota > 0 .$$

Definition 5 ([16]): Suppose $g \in C_{-1}^p$ and $p > 0$. Then the Caputo fractional derivative of order γ of function $g(t)$ is represented as

$$\frac{d^\gamma g(t)}{dt^\gamma} = \frac{1}{\Gamma(p-\gamma)} \int_0^t (t-\theta)^{p-\gamma-1} g^{(p)}(\theta) d\theta, p-1 < \gamma \leq p .$$

Let V be a Banach space. Consider $B(V)$ denote the collection of all bounded linear operators on V . When g is an abstract function taking values in V , the integrals and derivatives given in Definition 4 and 5 are taken in Bochner's sense.

To prove the exact controllability results in this paper, we apply the fixed point theorem, which serves as a fundamental tool, as follows:

Theorem 2 ([32]): "Let Z be a Banach space and let $E \subset Z$ be a convex closed set. Consider a relatively open subset $S \subset E$ with $0 \in S$. If $H : S \to E$ is a compact mapping, then one of the following alternative holds:

(1) There exists $u \in \partial S$ and a scalar $\Xi \in (0,1)$ in such a way that $u = \Xi H(u)$

(2) H has a fixed point in $\bar{S}$."

Definition 6 ([33]): A continuous mapping $y(.) : D \to V$ is referred to as a mild solution of Equations (1)-(2) and $\forall y_0 \in V$, it satisfies the corresponding integral equation

$$y(t) = N(t)\left[y_0 - \varsigma(y)\right] + \frac{1}{\Gamma\gamma}\int_0^t (t-\theta)^{\gamma-1} N(t-\theta) \times \left[\Upsilon x(\theta) + J\left(\theta, y(\theta), \int_0^\theta m_1(\theta,\Psi,y(\Psi))d\Psi, \int_0^c m_2(\theta,\Psi,y(\Psi))d\Psi\right)\right] d\theta, t \in D \quad (3)$$

Definition 7: The system described by Equations (1)-(2) is said to be controllable on the prescribed interval $[0,c]$, if $\forall y_0, y_1 \in V$ and there exists a control $x \in L^2(D,B)$ such that the corresponding mild solution $y(t)$ of Equations (1)-(2) satisfies the terminal condition $y(c) = y_1$.

4. Exact Controllability Results

To investigate the exact controllability results of Equations (1)-(2), we impose the following assumptions:

(A_1) **Nonlinearity J (Carathéodory condition).**

The function $J : D \times V \times V \times V \to V$ satisfy the Caratheodory condition, i.e.

For every $t \in D$, the mapping $J(t,.,.,.) : V \times V \times V \to V$ is continuous;
For every $x, y, z \in V$, the mapping $J(.,x,y,z) : D \to V$ is strongly measurable;

(A_2) **Volterra–Fredholm kernels.**

The functions $m_1, m_2 : \Delta \times V \to V$ satisfy the Caratheodory condition, i.e.

For every $(t,\theta) \in \Delta$, the function $m_1(t,\theta,.) : V \to V$ and $m_2(t,\theta,.) : V \to V$ are continuous;
For every $x \in V$, the function $m_1(.,.,x) : \Delta \to V$ and $m_2(.,.,x) : \Delta \to V$ are strongly measurable;

(A_3) **Uniform boundedness on bounded sets.**

For every positive integer r, there exists $\chi_r \in L^1(\varpi)$ such that for almost everywhere $t \in D$ and $y \in C(D,V)$

$$\sup_{\|y\| \le r} \left\| J\left(t, y(t), \int_0^t m_1(t,\theta,y(\theta))d\theta, \int_0^c m_2(t,\theta,y(\theta))d\theta\right) \right\| \le \chi_r(t),$$

where $\|y\| = \sup_{t \in D} \|y(t)\|$;

(A_4) **Growth bounds for m_1, m_2.**

There exist $n_1, n_2 \in L^1(D, R_+)$ such that

(a) $\|m_1(t,\theta,u)\| \le n_1(t)\upsilon(\|u\|)$, $(\varpi,\theta) \in \Delta, u \in V$,

where $\upsilon : [0,\infty) \to (0,\infty)$ is a continuous non-decreasing function;

(b) $\|m_2(t,\theta,v)\| \le n_2(t)\upsilon(\|u\|)$, $(t,\theta) \in \Delta, v \in V$,

where $\upsilon : [0,\infty) \to (0,\infty)$ is a continuous function which is non-decreasing;

(A_5) **Growth bounds for J.**

There exists $h \in L^1(D, R_+)$ such that

$\|J(t,u,v,w)\| \le h(t)\upsilon(\|u\| + \|v\| + \|w\|)$, for every $(t,u,v,w) \in D \times V \times V \times V$

(A_6) **Semigroup assumption.**

A is the infinitesimal generator of a Strongly continuous semigroup of bounded linear operators $N(t), t \ge 0$ in V, which is compact for $t > 0$, and there exist constant $F_1 \ge 1$ in such a manner that $\|N(t)\|_{\Upsilon(V)} \le F_1, t \ge 0$.

(A_7) **Nonlocal term.**

The function $\varsigma(.) : C(D,V) \to V$ is considered to be continuous. There exists a constant $F_2 > 0$ such that $\|\varsigma(y)\| \le F_2$ for any $y \in V$.

(A_8) **Control operator.**

The linear operator $Q : L^2(D,B) \to V$, given by

$$Qx = \frac{1}{\Gamma\gamma}\int_0^c (c-\theta)^{\gamma-1} N(c-\theta)\Upsilon x(\theta)d\theta .$$

has a bounded invertible operator $\tilde{Q}^{-1} : V \to L^2(D,B)$. Moreover, there exist $F_3 \ge 0$ and $F_4 \ge 0$ such that $\|\Upsilon\| \le F_3$, $\|\tilde{Q}^{-1}\| \le F_4$. (see [34])

(A_9) **A priori bound condition.**

There exists a constant $F^* > 0$ with

$$\frac{F^*}{M_1 + \left(M_2 + \frac{c^{\gamma-1}}{\Gamma(\gamma)}L^*\right)\upsilon(F^*)\int_0^c \{h(\theta) + n_1(\theta) + n_2(\theta)\}d\theta} > 1,$$

where

$$M_1 = L^*\left(\|y_0\| + F_2\right) + \frac{c^\gamma}{\Gamma(\gamma+1)}L^* F_3 F_4\left(\|y_1\| + L^*\left(\|y_0\| + F_2\right)\right)$$

and

$$M_2 = \frac{c^{2\gamma-1}}{\Gamma(\gamma)\Gamma(\gamma+1)}\left(L^*\right)^2 F_3 F_4 .$$

Remark 1:

(1) The procedure for constructing the bounded inverse operator $\tilde{Q}^{-1}$ in a general Banach space is described in [34]; therefore, its proof is omitted.
(2) If the space V is finite-dimensional, assumption (A_8) reduces to the requirement that the related Gramian matrix be invertible and positive definite (see [34]).
(3) For general Banach spaces, assumption (A_8) has been widely employed in the literature. (see [35,36])

Now we will prove our main theorem. The proof is based on fixed point approach. First, a suitable control function is constructed by employing assumption (A_8), through the inverse of the controllability operator, so that the terminal condition $y(c) = y_1$ is satisfied. The system is then rewritten as an equivalent fixed point problem in $C(D,V)$. Further, by applying the Carathéodory conditions on the nonlinear terms, the compactness of the semigroup, and the Arzelà–Ascoli theorem, the associated operator is shown to be completely continuous. Subsequently, appropriate a priori bounds are established to exclude solutions on the boundary of a closed ball in $C(D,V)$. Finally, the Leray–Schauder nonlinear alternative is used to obtain a fixed point, which ensures the controllability of the system.

Theorem 3: If the assumptions $(A_1)-(A_8)$ are satisfied. Then, the system described by Equations (1)-(2) is controllable on D.

Proof: In light of the assumption (A_8), for an arbitrary function $y(t)$, the control is defined as follows

$$x_y(t) = \tilde{Q}^{-1}\left[y_1 - N(c)(y_0 - \varsigma(y)) - \frac{1}{\Gamma(\gamma)}\int_0^c (c-\theta)^{\gamma-1} N(c-\theta) J\left(\theta, y(\theta), \int_0^\theta m_1(\theta,\Psi,y(\Psi))d\Psi, \int_0^c m_2(\theta,\Psi,y(\Psi))d\Psi\right)d\theta\right](t).$$

In the subsequent analysis, it is only required to confirm that with this control, the operator $\Phi : C(D,V) \to C(D,V)$ given in such a way that

$$\Phi y(t) = N(t)\left[y_0 - \varsigma(y)\right] + \frac{1}{\Gamma\gamma}\int_0^t (t-\theta)^{\gamma-1} N(t-\theta) \times \left[\Upsilon x(\theta) + J\left(\theta, y(\theta), \int_0^\theta m_1(\theta,\Psi,y(\Psi))d\Psi, \int_0^c m_2(\theta,\Psi,y(\Psi))d\Psi\right)\right] d\theta, t \in D ,$$

admits a fixed point $y(.)$. The existence of this fixed point implies that it represents a mild solution to Equations (1)-(2). Furthermore, since $\Phi y(c) = y_c$, we deduce that the system is controllable.

Following the approach used in [26], our first step is to prove that Φ possesses both continuity and complete continuity. To simplify our analysis, we assume that

$$U(\hbar) = \Upsilon \tilde{Q}^{-1}\left[y_1 - N(c)(y_0 - \varsigma(y)) - \frac{1}{\Gamma(\gamma)}\int_0^c (c-\theta)^{\gamma-1} N(c-\theta) \wp(\theta) d\theta\right](\hbar)$$

Where $\wp(\theta) = J\left(\theta, y(\theta), \int_0^\theta m_1(\theta,\Psi,y(\Psi))d\Psi, \int_0^c m_2(\theta,\Psi,y(\Psi))d\Psi\right)$.

For some $r \geq 1$, define $T_r = \left\{ y \in C(D,V) \middle| \|y\| \leq r \right\}$. Clearly, if $y \in T_r$, then

$$\|U(\hbar)\| \leq F_3 F_4\left[\|y_1\| + L^*(\|y_0\| + F_2) + \frac{c^{\gamma-1}}{\Gamma(\gamma)} L^* \int_0^c \chi_r(\theta) d\theta\right] \coloneqq U_0$$

and

$$\|\Phi y(t)\| \leq L^*\left[\|y_0\| + F_2\right] + \frac{c^\gamma}{\Gamma(\gamma+1)} L^* U_0 + \frac{c^{\gamma-1}}{\Gamma(\gamma)} L \|\chi_r\|_{L^1};$$

Given that χ_r satisfies the assumption (A_3), select an arbitrary $\lambda > 0$. Let $\kappa_1, \kappa_2 \in D$ be taken so that $\kappa_2 > \kappa_1$. We divide the analysis into two steps depending on the value of κ_1 relative to λ: either $\kappa_1 > \lambda$ and $\kappa_1 \leq \lambda$.

Step 1. When $\kappa_1 > \lambda$, then

$$\begin{aligned}
\|\Phi y(\kappa_2) - \Phi y(\kappa_1)\| &\leq \|N(\kappa_2)(y_0 - \varsigma(y)) - N(\kappa_1)(y_0 - \varsigma(y))\| + \frac{1}{\Gamma\gamma}\int_0^{\kappa_1-\lambda} \|(\kappa_2-\theta)^{\gamma-1} N(\kappa_2-\theta) \\
&- (\kappa_1-\theta)^{\gamma-1} N(\kappa_1-\theta)\| \|U(\theta) + \wp(\theta)\| d\theta + \frac{1}{\Gamma(\gamma)}\int_{\kappa_1-\lambda}^{\kappa_1} \|(\kappa_2-\theta)^{\gamma-1} N(\kappa_2-\theta) \\
&- (\kappa_1-\theta)^{\gamma-1} N(\kappa_1-\theta)\| \|U(\theta) + \wp(\theta)\| d\theta + \frac{1}{\Gamma(\gamma)}\int_{\kappa_1}^{\kappa_2} (\kappa_2-\theta)^{\gamma-1} \times \|N(\kappa_2-\theta)\| \|U(\theta) + \wp(\theta)\| d\theta \\
&\leq \|N(\kappa_2)(y_0) - N(\kappa_1)(y_0)\| + L^*\|N(\kappa_2) - N(\kappa_1)\| + \frac{c^{\gamma-1}}{\Gamma(\gamma)} L^*\left[\|(\kappa_2-\theta)^{\gamma-1} N(\kappa_2-\kappa_1+\lambda) - (\kappa_1-\theta)^{\gamma-1} N(\lambda)\| \int_0^{\kappa_1-\lambda} [U_0 + \chi_r(\theta)] d\theta\right] \\
&+ \frac{c^{\gamma-1}}{\Gamma(\gamma)} 2L^* \int_{\kappa_1-\lambda}^{\kappa_1} [U_0 + \chi_r(\theta)] d\theta + \frac{c^{\gamma-1}}{\Gamma(\gamma)} L^* \int_{\kappa_1}^{\kappa_2} [U_0 + \chi_r(\theta)] d\theta ,
\end{aligned}$$

where the following semigroup identities are applied:

$$N(\kappa_1 - \theta) = N(\kappa_1 - \theta - \lambda) N(\lambda)$$

and

$$N(\kappa_2 - \theta) = N(\kappa_2 - \kappa_1 + \lambda) N(\kappa_1 - \theta - \lambda).$$

Step 2. Now assume the case where $\kappa_1 \leq \lambda$. If, in addition, the difference $\kappa_2 - \kappa_1 < \lambda$, we get

$$\|\Phi y(\kappa_2) - \Phi y(\kappa_1)\| \leq \|N(\kappa_2)(y_0 - \varsigma(y)) - N(\kappa_1)(y_0 - \varsigma(y))\| + \frac{1}{\Gamma\gamma}\int_0^{\kappa_2} (\kappa_2-\theta)^{\gamma-1} \|N(\kappa_2-\theta)\| \times$$

$$\left\|U(\theta)+\wp(\theta)\right\|d\theta+\frac{1}{\Gamma(\gamma)}\int_0^{\kappa_1}(\kappa_1-\theta)^{\gamma-1}\left\|N(\kappa_1-\theta)\right\|\left\|U(\theta)+\wp(\theta)\right\|d\theta$$

$$\leq\left\|N(\kappa_2)(y_0)-N(\kappa_1)(y_0)\right\|+L^*\left\|N(\kappa_2)-N(\kappa_1)\right\| +\frac{c^{\gamma-1}}{\Gamma(\gamma)}L^*\int_0^{2ƛ}\left[U_0+\chi_r(\theta)\right]d\theta+\frac{c^{\gamma-1}}{\Gamma(\gamma)}L^*\int_0^{ƛ}\left[U_0+\chi_r(\theta)\right]d\theta\ .$$

To demonstrate that Φ is equicontinuous, we rely on three main observations:

(1) the semigroup is strongly continuous;
(2) Lemma 1;
(3) $\forall t>0$, the operator $N(t)$ which is compact and therefore continuous with respect to the uniform operator topology.

Now, fix any point t with $0<t\leq c$. Take a real number $ƛ$ in such a manner that $0<ƛ<t$. For every $y\in T_r$, we introduce the following:

$$\Phi_{ƛ}y(t)=N(t)\left[y_0-\varsigma(y)\right]+\frac{1}{\Gamma\gamma}\int_0^{t-ƛ}(t-\theta)^{\gamma-1}N(t-\theta)\left[U(\theta)+\wp(\theta)\right]d\theta$$

$$=N(t)\left[y_0-\varsigma(y)\right]+\frac{N(ƛ)}{\Gamma\gamma}\int_0^{t-ƛ}(t-\theta)^{\gamma-1}N(t-\theta-ƛ)\left[U(\theta)+\wp(\theta)\right]d\theta\ .$$

It should be taken into account that

$$\left\|\int_0^{t-ƛ}N(t-\theta-ƛ)\left[U(\theta)+\wp(\theta)\right]d\theta\right\|\leq L^*\int_0^{t-ƛ}\left[U(0)+\chi_r(\theta)\right]d\theta$$

It is given that, for $t>0$, $N(t)$ is a compact operator. The set $X_{ƛ}(t)=\left\{\Phi_{ƛ}y(t)\,\middle|\,y\in T_r\right\}$ is relatively compact in V for each $ƛ$, $0<ƛ<t$.

Additionally, $\forall y\in T_r$ we get

$$\left\|\Phi y(t)-\Phi_{ƛ}y(t)\right\|\leq\frac{c^{\gamma-1}}{\Gamma(\gamma)}L^*\int_{t-ƛ}^{ƛ}\left[U_0+\chi_r(\theta)\right]d\theta\ .$$

This implies that the set

$X(t)=\left\{\Phi y(t)\,\middle|\,y\in T_r\right\}$ is totally bounded and hence relatively compact in V. Employing the Arzelà–Ascoli theorem, we conclude that $\Phi:C(D,V)\to C(D,V)$ is completely continuous.

Suppose a sequence $\{y_n\}_0^\infty\subseteq C(N,V)$ in such a manner that $y_j\to y$ in $C(D,V)$. Then, there exists an integer r satisfying $\left\|y_j(t)\right\|\leq r$, $\forall j\in\mathbb{N}$ and $t\in D$. Consequently, $y_j\in T_r$ and $y\in T_r$. For convenience, we set

$$U_j(\hbar)=\Upsilon\tilde{Q}^{-1}\left[y_1-N(c)\left(y_0-\varsigma(y)\right)-\frac{1}{\Gamma(\gamma)}\int_0^c(c-\theta)^{\gamma-1}N(c-\theta)\wp_j(\theta)d\theta\right](\hbar)\ ,$$

where $\wp_j(\theta)=J\left(\theta,y_j(\theta),\int_0^\theta m_1\left(\theta,\Psi,y_j(\Psi)\right)d\Psi,\int_0^c m_2\left(\theta,\Psi,y_j(\Psi)\right)d\Psi\right)$.

With the use of (A_1) and (A_2),

$\wp_j(t)\to\wp(t)$ when $j\to\infty$,

$\forall t\in D$, therefore

$\left\|\wp_j(t)-\wp(t)\right\|\leq 2\chi_r(t)$

and

$$\left\|\Phi y_j(t)-\Phi y(t)\right\|\leq\frac{c^{\gamma-1}}{\Gamma(\gamma)}L^*\int_0^t\left\|U_j(\theta)-U(\theta)\right\|d\theta+\frac{c^{\gamma-1}}{\Gamma(\gamma)}L^*\int_0^t\left\|\wp_j(\theta)-\wp(\theta)\right\|d\theta\ ,$$

where

$$\left\|U_j(\theta)-U(\theta)\right\| \le F_3F_4\frac{c^{\gamma-1}}{\Gamma(\gamma)}L^*\int_0^c\left\|\wp_j(\theta)-\wp(\theta)\right\|d\theta ,$$

With the use of dominated convergence theorem, it follows that

$\left\|\Phi y_j-\Phi y\right\|\to 0$ when $j\to\infty$.

Thus, the operator Φ demonstrates both continuity and complete continuity.

Now choose $0<\Xi<1$ and consider that y satisfy the relation $y=\Xi\Phi y$, then $\forall t\in D$, we obtain

$$y(t)=\Xi N(t)\left[y_0-\varsigma(y)\right]+\frac{\Xi}{\Gamma\gamma}\int_0^t(t-\theta)^{\gamma-1}N(t-\theta)\Gamma x_y(\theta)d\theta+\frac{\Xi}{\Gamma\gamma}\int_0^t(t-\theta)^{\gamma-1}N(t-\theta)\wp(\theta)d\theta, t\in D$$

where

$$x_y(\theta)=\tilde{Q}^{-1}\left[y_1-N(c)\left(y_0-\varsigma(y)\right)-\frac{1}{\Gamma(\gamma)}\int_0^c(c-\theta)^{\gamma-1}N(c-\theta)\wp_j(\theta)d\theta\right](\theta)$$

Furthermore, we get

$$\left\|y(t)\right\|=\left\|\Xi N(t)\left(y_0-\varsigma(y)\right)+\frac{\Xi}{\Gamma\gamma}\int_0^t(t-\theta)^{\gamma-1}N(t-\theta)\Gamma x_y(\theta)d\theta\right.$$

$$\left.+\frac{\Xi}{\Gamma\gamma}\int_0^t(t-\theta)^{\gamma-1}N(t-\theta)\wp(\theta)d\theta\right\|$$

$$\le L^*\left(\left\|y_0\right\|+F_2\right)+\frac{c^\gamma}{\Gamma(\gamma+1)}L^*F_3F_4\left[\left\|y_1\right\|+L^*\left(\left\|y_0\right\|+F_2\right)+\frac{c^{\gamma-1}}{\Gamma(\gamma)}L^*\times\right.$$

$$\left.\int_0^c\left\{h(\theta)+n_1(\theta)+n_2(\theta)\right\}\upsilon\left(\left\|y(\theta)\right\|\right)d\theta\right]+\frac{c^{\gamma-1}}{\Gamma(\gamma)}L^*\int_0^\varpi\left\{h(\theta)+n_1(\theta)+n_2(\theta)\right\}\upsilon\left(\left\|y(\theta)\right\|\right)d\theta$$

$$\le M_1+M_2\upsilon\left(\left\|y\right\|\right)\int_0^c\left\{h(\theta)+n_1(\theta)+n_2(\theta)\right\}d\theta+\frac{c^{\gamma-1}}{\Gamma(\gamma)}L^*\upsilon\left(\left\|y(\theta)\right\|\right)\int_0^\varpi\left\{h(\theta)+n_1(\theta)+n_2(\theta)\right\}d\theta .$$

Thus,

$$\frac{\left\|y\right\|}{M_1+\left(M_2+\frac{c^{\gamma-1}}{\Gamma(\gamma)}L^*\right)\upsilon\left(\left\|y\right\|\right)\int_0^c\left\{h(\theta)+n_1(\theta)+n_2(\theta)\right\}d\theta}\le 1 .$$

Using assumption (A_9) , there exists a positive constant F^* such that $\left\|y\right\|\ne F^*$.

Finally, the set

$$S\in\left\{y\in C(D,V)\middle|\left\|y\right\|<F^*\right\}$$

With the choice of S there exists no $y\in\partial S$ satisfying $y=\Xi\Phi y$ for any $0<\Xi<1$. Consequently, employing Theorem 2, we conclude that Φ has at least one fixed point y in S . This guarantees that the Equations (1)-(2) are controllable on D . Consequently, the system described by Equations (1)-(2) achieves exact controllability over the domain.

5. Examples

Example 1: Here, we provide a concrete example to illustrate the application and validity of the principal outcomes established in this study. Nonlinear Volterra-Fredholm integro-differential equations are of significant interest because such equations model a wide range of natural phenomena in which the system’s future evolution is governed by both its current state and its historical dynamics.

These equations generally typically arise in fields such as engineering, physics, control theory, and biology, particularly in systems with delayed interactions, memory effects, or hereditary properties. Assume the fractional heat equation with nonlinear memory and distributed control of the form:

$$\frac{\partial^\gamma}{\partial t^\gamma}\alpha(t,z^*)=\frac{\partial^2}{\partial z^{*2}}\alpha(t,z^*)+\Lambda(t,z^*)+\xi\left(t,\alpha(t,z^*),\int_0^t \ell_1(t,\theta,\alpha(\theta,z^*))d\theta,\int_0^c \ell_2(t,\theta,\alpha(\theta,z^*))d\theta\right), \quad (4)$$

$$\alpha(0,z^*)+\int_0^1 e(\theta)\alpha(\theta,z^*)d\theta=\alpha_0(z^*),0<z^*<\pi, \quad (5)$$

$$\alpha(t,0)=\alpha(t,\pi),t\in D=[0,1]. \quad (6)$$

Here, the fractional order $\gamma\in(0,1)$ and the mapping $\Lambda:D\times(0,\pi)\to(0,\pi)$ are continuous.

We assume the state and control spaces as

$V=U=L^2[0,\pi]$

and the control input is defined as $\Upsilon x(t)=\Lambda(t,.)$. Consider the linear operator $A:V\to V$ and given by $A\mu=\frac{d^2\mu}{dz^{*2}},\mu\in\text{dom}(A)$.

Its domain is given by

$\text{dom}(A)=\{\mu\in V:\mu,\frac{d\mu}{dz^*}$ are absolutely continuous, $\frac{d^2\mu}{dz^{*2}}\in V,\mu(0)=\mu(\pi)=0\}$

The operator A can be represented by an eigen function expansion of the following form

$A\mu=\sum_{n=1}^{\infty}n^2\langle\mu,\mu_n\rangle\mu_n,\mu\in\text{dom}(A)$,

in which the normalized eigen functions $(\mu_n)_{n\geq1}$ are defined as

$\mu_n(z^*)=\sqrt{\frac{2}{\pi}}\sin nz^*,n=1,2,\ldots$.

These functions form an orthonormal basis of V.

Consequently, following the ideas of semigroup theory, that A generates an analytic semigroup $\{N(t)\}_{t>0}$ on V which is clearly represented as

$N(t)\mu=\sum_{n=1}^{\infty}e^{-n^2t}\langle\mu,\mu_n\rangle\mu_n,\mu\in V$.

Further, the semigroup $N(t)$ satisfies the assumption (A_6).

Moreover, consider the function $a,b:\Delta\times R\to R,\Delta=\{t,\theta|0\leq\theta\leq t\leq1\}$ and $\xi:D\times R\times R\times R\to R$ are continuous and that the functions a,b,ξ satisfy the assumptions $(A_1)-(A_5)$.

Consider $\Upsilon x(t)=\Lambda(t,z^*),\chi=I$, where I is the identity operator is.

Define

$J(t,u,v,w)(z^*)=\xi(t,u(z^*),v(z^*),w(z^*))$

$m_1(t,\theta,u)(z^*)=a(t,\theta,u(z^*))$

$m_2(t,\theta,u)(z^*)=b(t,\theta,u(z^*))$,

$\zeta(u)(z^*)=\int_0^1 e(\theta)\alpha(\theta,z^*)d\theta$.

Then, Equations (1)-(2) is the abstract formulation of (4)-(6).

Now, let us define the operator $Q:L^2(D,B)\to V$ as follows

$Qx=\frac{1}{\Gamma\gamma}\sum_{n=1}^{\infty}\int_0^1(1-\theta)^{\gamma-1}e^{-n^2(1-\theta)}(x(\theta),\mu_n)\mu_n d\theta$.

Suppose that Q is invertible and satisfy assumption (A_8). With these assumptions, all the requirements of Theorem 3 are fulfilled. Consequently, the system described by Equations (4)-(6) admits exact controllability over the domain D.

Remark 2:

Here we will specially verify boundedness, compactness, and invertibility of the operator Q exactly as per required in Assumption (A_8) of this manuscript.

Verification:

Boundedness of the operator Q

From the Assumption (A_8), the operator $Q : L^2(D,B) \to V$ is given by

$$(Qx) = \frac{1}{\Gamma\gamma}\int_0^c (c-\theta)^{\gamma-1} N(c-\theta)\Upsilon x(\theta)d\theta .$$

From assumption (A_6), the semigroup $\{N(t)\}_{t\geq 0}$ which is generated by A is uniformly bounded, that is, $\|N(t)\| \leq F_1$, $t \geq 0$.

Also, the control operator Υ is bounded with $\|\Upsilon\| \leq F_3$.

Hence, $\forall x \in L^2(D,B)$

$$\|Qx\| \leq \frac{1}{\Gamma\gamma}\int_0^c (c-\theta)^{\gamma-1}\|N(c-\theta)\|\|\Upsilon\|\|x(\theta)\|d\theta$$

$$\leq \frac{F_1F_3}{\Gamma\gamma}\int_0^c (c-\theta)^{\gamma-1}\|x(\theta)\|d\theta .$$

So, there exists a positive constant $C > 0$ in such a way that $\|Qx\| \leq C\|x\|_{L^2(D,B)}$,

which demonstrate that Q is a bounded linear operator which satisfy the boundedness requirement in (A_8).

Compactness of the operator Q

From Assumption (A_6), the semigroup $N(t)$ is compact $\forall t > 0$. Since Υ is bounded, the composition $N(c-\theta)\Upsilon : B \to V$ is compact $\forall \theta \in (0,c]$.

The operator Q is represented as a Bochner integral of compact operators over a finite interval:

$$Q = \frac{1}{\Gamma\gamma}\int_0^c (c-\theta)^{\gamma-1} N(c-\theta)\Upsilon d\theta$$

Thus, from standard properties of compact operators in Banach spaces, Q is compact on V.

Invertibility of the operator Q

Assumption (A_8) clearly states that (i) Q is bijective, and (ii) its inverse $\widetilde{Q}^{-1} : V \to L^2(D,B)$ is bounded, that is, $\left\|\widetilde{Q}^{-1}\right\| \leq F_4$

This assumption is verified in the paper by referring to the construction in [34], where the controllability operator is demonstrated to be invertible via pseudo-inverse techniques. So, we omit the detailed proof here. Consequently, Assumption (A_8) is fully satisfied.

Example 2: Here, we give another example to illustrate the usefulness of our main result. Let us consider the following partial integro-differential equation of the form

$$\frac{\partial^\gamma}{\partial t^\gamma}\alpha(t,z^*) = \frac{\partial^2}{\partial z^{*2}}\alpha(t,z^*) + \Lambda(t,z^*) + \alpha(t,z^*) + \int_0^t \frac{1}{(1+t^2)(1+\theta)}\alpha(\theta,z^*)d\theta + \int_0^1 \frac{1}{(1+t^2)(1+\theta)}\left[\alpha^2(\theta,z^*) + \sin\alpha^2(\theta,z^*)\right]d\theta \tag{7}$$

$$\alpha(t,0) = \alpha(t,\pi), t \in [0,1] \tag{8}$$

$$\alpha(0,z^*) + \int_0^1 \frac{1}{2}\alpha(\theta,z^*)d\theta = \alpha_0(z^*), 0 \leq t \leq 1, 0 \leq z^* \leq \pi \tag{9}$$

under the conditions as we have discussed in Example 1, the Equations (7)-(9) can be reformulate in abstract form as given in Equations (1)-(2).

6. Conclusion

In the present work, we explore the exact controllability of a class of nonlinear Volterra–Fredholm integro-differential equations (NNVFFIE) in Banach spaces. We establish verifiable conditions guaranteeing the exact controllability of the system through an appropriately constructed control operator by employing semigroup theory in conjunction with fractional calculus. The analysis relies on compactness arguments and fixed-point techniques, yielding exact controllability criteria under mild growth and regularity assumptions on the nonlinear terms. The derived results advance the existing literature by increasing known exact controllability conditions from classical and integer-order systems to fractional-order models with integral-type nonlinearities.

The contributions of the present paper are twofold. Firstly, the abstract framework demonstrates that fractional systems with hereditary effects are controllable under relatively weak assumptions. Secondly, the theoretical results are illustrated with an example, highlighting the applicability of the abstract claims to fractional partial differential equations driven by semigroup-generating operators. The proposed framework also suggests future research directions, including approximate and robust controllability, constrained control problems, stochastic and impulsive extensions, and numerical and optimal control formulations for fractional systems.

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Conflict of Interest

The author confirms that there are no known financial or personal conflicts of interest that could have influenced the findings or interpretations presented in this study.

Data Availability Statement

No datasets were generated or analyzed during the current study. All results are based on theoretical analysis.

Generative AI Statement

The authors declare that no Generative AI was used in the creation of this manuscript.

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