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A Localized Fixed Point Framework for Uncertain Dynamic Programming in Fuzzy S-Metric Spaces

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Abstract

In this paper, a local fixed point theorem for α-contractive mappings in fuzzy S-metric spaces is proved. The result is stated on a closed ball and assumes a localized contractive condition along with an admissibility condition at the beginning. These conditions ensure that the closed ball is mapped to itself by the given function and the iterative sequence related to the function converges to a unique fixed point in the same environment. The localized contractive condition allows the fixed point theorem to be applied in contexts where the contractive condition might not be valid, thus broadening the applications of fixed point theory in fuzzy S-metric spaces. To show the efficiency and applicability of the proposed outcomes of the theory, a real-life problem that occurs in the value iteration process in dynamic programming is provided in the following scenario. In this problem, the fuzzy S-metric theory is capable of incorporating the uncertainties and imprecise concepts of the reward function or transition mechanism, usually found in real-life problems of this category. The results attained within this study offer a robust basis for analysis on the convergence properties of computational methods, optimization processes, and iterative algorithms under uncertainty conditions. In addition, it further enhances the advancements of fixed points theory within fuzzy S-metric spaces and provides new avenues for applications within the areas of applied mathematics and related disciplines.

Keywords

Fuzzy S-metric space, Local fixed point theorem, α-contractive mapping, Admissible mapping, Value iteration, Dynamic programming, Convergence analysis, Uncertainty modeling

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1. Introduction

In many real-world scenarios, it can be difficult to determine the distances between objects that cannot be measured precisely. The use of suitable mathematical tools that can capture such imprecision is required due to this difficulty. As a result, a number of generalized distance concepts have been proposed to increase fixed point theory's applicability. Both the multiplicity and uniqueness of fixed points have been studied under various metric frameworks within this theory, resulting in significant theoretical advancement. In many branches of mathematics, such as analysis, topology, and applied mathematics, metric spaces serve as a fundamental framework. Numerous generalizations of metric spaces have been proposed by researchers, driven by both theoretical and practical considerations. These generalizations have resulted in new fixed point results. In this regard, Dhage offered D-metric spaces [1], and Göhler introduced the idea of 2-metric spaces [2]. Subsequent research, however, identified certain flaws in these formulations (see [3,4]). Sedghi et al. established the concept of an S-metric space in 2012 [5] to solve these problems, demonstrating that it is a true extension of conventional metric spaces. They also looked into the basic characteristics of S-metric spaces and formulated a number of fixed point theorems. Subsequently, Sedghi and Dung found more fixed point solutions in S-metric spaces by further extending this framework [6]. Lotfi A. Zadeh formally proposed the concept of fuzziness in 1965 through the theory of fuzzy sets [7], which encapsulates ambiguity and uncertainty. Unlike classical sets, fuzzy sets allow elements to belong to a set with varying degrees of membership, which makes them useful for modeling imprecise information. This idea inspired researchers to develop fuzzy metric spaces, which treat distance as a fuzzy variable rather than a precise numerical value. In this context, the proximity between points is defined by degrees of nearness, which reflect the uncertainty inherent in many real-world problems. In 1975, Kramosil and Michałk developed the fundamental theory of a fuzzy metric space based on these concepts [8]. George and Veeramani later published a revised definition based on continuous t-norms, providing a more structured and widely accepted framework [9]. Their formulation generalized the work of Kaleva and Seikkala [10], and later by Kramosil and Michěk [8]. A fuzzy variant of the Banach fixed point theorem initiated the exploration of fixed points applied to fuzzy metric spaces by Grabiec [11]. Statistical metric spaces, which incorporate probabilistic notions of distance, were introduced by Berthold Schweizer et al. [12] as a generalization of classical metric spaces. Move over, fixed point methods are applied to the design of optical structures to optimize the performance of transmitted waves and ensure stable propagation applied to the design of optical structures of lightweight materials [13]. Furthermore, fixed point methods are applied to information transmission models to increase transmission efficiency and correct errors in transmission detection systems applied to information transmission models [14]. Contraction kind mappings are utilized to detect eco-signal patterns from background noises in eco-signal detection problems [15]. This enabled the rise of other pivotal ideas, like the notion of convergence, Cauchy sequences, and the notion of completeness applied to fuzzy metrics. Since fixed point techniques gained popularity and are currently applied to various branches of science and engineering. A Gaussian Decomposition Algorithm [16], to cite one, applies iterative methods that converge using fixed points. The fuzzy S-metric space was introduced in [17], where its fundamental properties and related fixed point results were discussed.

Many modern computational models utilize iterative schemes in order to support stable operating points. Iterative schemes occur naturally in many computer science domains, for example, in optimization techniques, parallel computing, machine learning, and decision support systems. Stability analysis, which guarantees the convergence of these iterative schemes to unique fixation points, represents an important principle of these computer systems[18,19]. Classical convergence analysis can often be developed inside classical metric spaces. In real-life computations, though, uncertainties, approximations, and tolerance levels might influence the process, as exact distance computation can fail to capture the actual dynamics of the system with sufficient resolution. The key results generalize the idea of fixed points from classical calculus to fuzzy structures, establishing the existence of fixed points for fuzzy contractive mappings [20-22]. This has triggered interest in fuzzy mathematical structures like fuzzy and fuzzy S-metric spaces, providing a framework with greater flexibility in addressing imprecision in iterative computer environments [23,24]. The theory of fixed points in fuzzy metric structures has also gained extensive literature coverage, establishing its efficacy in the analysis of stability properties for computational procedures. A deeper comprehension of convergence processes in algorithms operating under partial knowledge or uncertainty has been made possible by these advances [25]. In many practical computational applications, however, assuming global contractive-ness of an update operator is often unrealistic. Instead, convergence typically occurs in a local sense, once the iterative process enters a sufficiently small neighborhood of a reference configuration. This observation has led to increased attention to local fixed point results, where contraction conditions are imposed only on restricted regions such as closed balls [26,27]. In computer science applications, where iterative procedures are intended to settle after a limited number of refining steps, such localized approaches are especially well-suited.

First presented by [17], fuzzy S-metric spaces offer a flexible framework for examining iterative mappings under uncertainty. In this framework, fixed point theory formally characterizes the stable states to which iterative processes converge. In this paper, we present local fixed point solutions for α-contractive mappings constructed on closed balls in fuzzy S-metric spaces. This is consistent with real-world computational settings where convergence often depends on gaining access to a suitable neighborhood. We demonstrate the existence and uniqueness of a fixed point by verifying both a localized contractive condition and an admissibility criterion of an initial type. We illustrate the relevance of these theoretical developments using a practical application to value iteration in dynamic programming, in which the fuzzy metric naturally models uncertainty in rewards or transitions. These results offer a solid analytical basis for studying the convergence problems encountered in iterative algorithms, optimization processes, and computational methodologies in general where imprecision may be inherent. They also broaden the application of fixed point theory in fuzzy S-metric spaces.

2. Preliminaries

This section outlines foundational concepts drawn from the existing literature.

Definition 1. ([18]) A binary operation $\triangle:[0,1]\times[0,1]\to[0,1]$ is called a continuous t-norm (ctn) if it is associative, commutative, continuous, and satisfies the boundary condition: $\zeta_0\triangle 1=\zeta_0 \triangle \zeta_0\triangle[0,1]$, and $\triangle$ is monotonic i.e., $\zeta_0\triangle\eta_0\le\gamma_0\triangle\vartheta_0$, whenever $\zeta_0\le\gamma_0$ and $\gamma_0\le\vartheta_0$, for each $\zeta_0,\eta_0,\gamma_0,\vartheta_0\in[0,1]$.

Definition 2. ([15]) Let Ω be a non-empty set. A function $S:\Omega\times\Omega\times\Omega\to[0,\infty)$ is called an S-metric on Ω, if, for all $\zeta_0,\eta_0,\gamma_0,a \in \Omega$, the following conditions are satisfied:

$S(\zeta_0,\eta_0,\gamma_0)\ge 0$,

$S(\zeta_0,\eta_0,\gamma_0)=0 \Leftrightarrow \zeta_0=\eta_0=\gamma_0$,

$S(\zeta_0,\eta_0,\gamma_0)\le S(\zeta_0,\zeta_0,a))+S(\eta_0,\eta_0,a)+S(\gamma_0,\gamma_0,a)$.

The pair (Ω,S) is called an S-metric space.

Definition 3. ([17]) The triple $(\Omega,F_S,\triangle)$ is said to be a fuzzy S-metric space if Ω is an arbitrary non-empty, set, $\triangle$ is a ctn, and $F_S:\Omega^3\times(0,\infty)\to[0,1]$ is a fuzzy set satisfying the following conditions for all $\zeta_0,\eta_0,\gamma_0,a \in \Omega$ and all s,t>0:

$F_S(\zeta_0,\eta_0,\gamma_0,\theta)>0$,

$F_S(\zeta_0,\eta_0,\gamma_0,\theta)=1 \;\forall\; \theta>0 \Leftrightarrow \zeta_0=\eta_0=\gamma_0$,

$F_S(\zeta_0,\eta_0,\gamma_0,\theta+u+v)\ge F_S(\zeta_0,\zeta_0,a,\theta)\triangle F_S(\eta_0,\eta_0,a,u)\triangle F_S(\gamma_0,\gamma_0,a,v)$.

$F_S(\zeta_0,\eta_0,\gamma_0,.):(0,\infty)\to[0,1]$ is left continuous and non-decreasing for all $\zeta_0,\eta_0,\gamma_0,a \in \Omega$ and $\theta,u,v>0$.

Definition 4. ([17]) A sequence $\{\zeta_n\}$ in a fuzzy S-metric space converges to $\zeta_0\in\Omega$, if

$\lim_{n\to\infty}F_S(\zeta_n,\zeta_n,\zeta_0,\theta)=1$,

for each $\theta>0$.

A sequence $\{\zeta_n\}$ in a fuzzy S-metric space is Cauchy if $\lim_{n\to\infty}F_S(\zeta_n,\zeta_n,\zeta_{n+m},\theta)=1$ for each $\theta>0$ and $m>0$.

A fuzzy S-metric space is called complete if every Cauchy sequence in it is convergent.

Definition 5. ([17]) Let $(\Omega,F_S,\triangle)$ be a fuzzy S-metric space. We define an open ball $B_S(\zeta_0,r,\theta)$ with center $\zeta_0\in\Omega$ and radius r, $0<r<1$ and $\theta>0$ as $B_S(\zeta_0,r,\theta)=\{\eta_0 \in \Omega:F_S(\eta_0,\eta_0,\zeta_0,\theta)>1-r\}$.

Proposition 1. ([17]) In a fuzzy S-metric space, every open ball is an open set.

3. Main Results

In this section, we present and prove several key properties of the space introduced earlier. We establish conditions under which certain mappings exhibit fixed points, and we demonstrate the relationships between different topological structures within this framework. These findings extend and unify several known results from the literature.

Definition 6. Let $(\Omega,F_S,\triangle)$ be a fuzzy S-metric space. A mapping $L:\Omega\to\Omega$ is said to be θ-uniformly continuous if for every ε with $0<\varepsilon<1$, $\exists$ δ with $0<\delta<1$ s.t.,

$F_S(\zeta_0,\zeta_0,\eta_0,\theta)\ge 1-\delta$ implies $F_S(L\zeta_0,L\zeta_0,L\eta_0,\theta)\ge 1-\varepsilon$, for all $\zeta_0,\eta_0\in\Omega$ and for all $\theta>0$.

Result 1. Let $(\Omega,F_S,\triangle)$ be a fuzzy S-metric space and let $L:\Omega\to\Omega$ be θ-uniformly continuous on Ω.

If $\{\zeta_n\}$ is a sequence in Ω such that $\zeta_n\to\zeta_0$ as $n\to\infty$ in $(\Omega,F_S,\triangle)$, then

$L(\zeta_n)\to L(\zeta_0)$ as $n\to\infty$ in $(\Omega,F_S,\triangle)$.

Proof.

Let $\zeta_n\to\zeta_0$ as $n\to\infty$ in $(\Omega,F_S,\triangle)$. By the definition of convergence, for every $\varepsilon\in(0,1)$ and for every $\theta>0$, $\exists$ $n_0\in N$ such that

$F_S(\zeta_n,\zeta_n,\zeta_0,\theta)\ge 1-\varepsilon$ for all $n\ge n_0$.

Since L is θ-uniformly continuous on Ω, corresponding to $\varepsilon \in (0,1)$, $\exists\ \delta \in (0,1)$ s.t.,

$F_S(\zeta_0,\zeta_0,\eta_0,\theta)\geq 1-\delta \Rightarrow F_S(L\zeta_0,L\zeta_0,L\eta_0,\theta)\geq 1-\varepsilon$,

for all $\zeta_0,\eta_0 \in \Omega$ and for all $\theta>0$.

Therefore,

for all $n\geq n_0$ and for all $\theta>0$,

we have

$F_S(L\zeta_n,L\zeta_n,L\zeta_0,\theta)\geq 1-\varepsilon$.

Hence, $L(\zeta_n)\to L(\zeta_0)$ as $n\to\infty$ in $(\Omega,F_S,\triangle)$.

Definition 7. Let $(\Omega,F_S,\triangle)$ be a fuzzy S-metric space. A mapping $L:\Omega\to\Omega$ is called a α-fuzzy contractive mapping if there exists a constant $\alpha \in (0,1)$ such that

$\alpha F_S(L(\zeta_0),L(\eta_0),L(\gamma_0),\theta)\geq F_S(\zeta_0,\eta_0,\gamma_0,\theta)$,

for all $\zeta_0,\eta_0,\gamma_0 \in \Omega$ with $\zeta_0\neq\eta_0\neq\gamma_0$ and for all $\theta>0$.

The above definition is a natural extension of the classical Banach contraction principle to fuzzy S-metric spaces. In classical metric spaces, contractions decrease distances, whereas in the fuzzy setting convergence is characterized by membership values approaching 1. The condition

$\alpha F_S(L(\zeta),L(\eta),L(\gamma),\theta)\geq F_S(\zeta,\eta,\gamma,\theta)$, $\alpha \in (0,1)$,

ensures that the image under L is strictly closer in the fuzzy sense. Moreover, when the F_S-metric reduces to a standard fuzzy metric, this condition coincides with known fuzzy contraction definitions, and for $\alpha=1$ it reduces to a non-expansive type mapping.

Theorem 1. In a fuzzy S-metric space $(\Omega,F_S,\triangle)$, any sequence that converges is also a Cauchy sequence.

Proof.

Let $\{\zeta_n\}$ be a convergent sequence in fuzzy S-metric space $(\Omega,F_S,\triangle)$. Then there exists $\zeta_0\in\Omega$ such that $\zeta_n\to\zeta_0$ as $n\to\infty$.

Let $\kappa \in (0,1)$ and $\theta>0$. Since $\zeta_n\to\zeta_0$, there exists $n_0\in N$ such that

$F_S(\zeta_n,\zeta_n,\zeta_0,\theta)\geq 1-\kappa$for all $n\geq n_0$,

and

$F_S(\zeta_m,\zeta_m,\zeta_0,\theta)\geq 1-\kappa$for all $n\geq n_0$.

By Definition 3 (property (3) of the fuzzy S-metric), we have

$$F_S(\zeta_n,\zeta_n,\zeta_m,\theta)\&\geq F_S(\zeta_n,\zeta_n,\zeta_0,\tfrac{\theta}{2})\triangle F_S(\zeta_0,\zeta_0,\zeta_m,\tfrac{\theta}{2})$$
$$>(1-\kappa)\triangle(1-\kappa)$$
$$\&=\min\{1-\kappa, 1-\kappa\}$$
$$\&=1-\kappa,$$

for all $n, m\geq n_0$.

Hence,

$F_S(\zeta_n,\zeta_n,\zeta_m,\theta)\geq 1-\kappa$for all $n,m\geq n_0$.

Therefore,

$\{\zeta_n\}$ is a Cauchy sequence in $(\Omega,F_S,\triangle)$.

Theorem 2. In a fuzzy S-metric space $(\Omega,F_S,\triangle)$, the limit of a convergent sequence is unique.

Proof.

Let $\{\zeta_n\}$ be a sequence in the fuzzy S-metric space $(\Omega,F_S,\triangle)$ which converges to two points $\zeta_0,\eta_0\in\Omega$. Assume that $\zeta_0\neq\eta_0$.

Since $\zeta_n\to\zeta_0$ and $\zeta_n\to\eta_0$, we have

$$\lim_{n\to\infty} F_S(\zeta_n,\zeta_n,\zeta_0,\theta)=1, \qquad (1)$$

and

$$\lim_{n\to\infty} F_S(\zeta_n,\zeta_n,\eta_0,\theta)=1, \qquad (2)$$

for all $\theta>0$.

By the definition of convergence, for any $\varepsilon \in (0,1]$ and $\theta>0$, there exist $n_1,n_2 \in Z^+$ such that

$F_S(\zeta_n,\zeta_n,\zeta_0,\theta)>1-\varepsilon$for all $n\geq n_1$,

and

$F_S(\zeta_n,\zeta_n,\eta_0,\theta))>1-\varepsilon$for all $n\geq n_2$.

Let $n_0=\min\{n_1,n_2\}$ and choose $n\geq n_0$. By Definition 3, property (3),

we obtain

$$F_S(\zeta_0,\zeta_0,\eta_0,\theta)\&\geq F_S(\zeta_0,\zeta_0,\zeta_n,\frac{\theta}{2})\triangle F_S(\zeta_n,\zeta_n,\eta_0,\frac{\theta}{2})$$
$$>(1-\varepsilon)\triangle(1-\varepsilon)$$
$$\&=\min\{1-\varepsilon,1-\varepsilon\}$$
$$\&=1-\varepsilon.$$

Thus,

$F_S(\zeta_0,\zeta_0,\eta_0,\theta)\geq 1-\varepsilon$.

Since ε is arbitrary, it follows that

$\lim_{\theta>0} F_S(\zeta_0,\zeta_0,\eta_0,\theta)=1$.

By Definition 3 of the fuzzy S-metric, this implies that $\zeta_0=\eta_0$.

Hence, the limit of a convergent sequence in a fuzzy S-metric space is unique.

Proposition 2. Let $(\Omega,F_S,\triangle)$ be a fuzzy S-metric space. If there exists a constant $k \in (0,1)$ such that

$F_S(\zeta_0,\zeta_0,\eta_0,k\theta)\geq F_S(\zeta_0,\zeta_0,\eta_0,\theta)$,

for all $\zeta_0,\eta_0 \in \Omega$ and for all $\theta>0$, then $\zeta_0=\eta_0$.

Proof.

Assume that

$F_S(\zeta_0,\zeta_0,\eta_0,k\theta)\geq F_S(\zeta_0,\zeta_0,\eta_0,\theta)$,

for all $\zeta_0,\eta_0 \in \Omega$ and $\theta>0$.

Then,

$$F_S(\zeta_0,\zeta_0,\eta_0,\theta)\&\geq F_S(\zeta_0,\zeta_0,\eta_0,\frac{\theta}{k})$$
$$\&\geq F_S(\zeta_0,\zeta_0,\eta_0,\frac{\theta}{k^2})$$
$$\&\geq F_S(\zeta_0,\zeta_0,\eta_0,\frac{\theta}{k^3})$$
$$\&$$
$$\&\geq F_S(\zeta_0,\zeta_0,\eta_0,\frac{\theta}{k^n}),$$

for all $n\in N$.

Since $k \in (0,1)$, we have $\frac{\theta}{k^n}\to\infty$ as $n\to\infty$. It follows that

$\lim_{n\to\infty} F_S(\zeta_0,\zeta_0,\eta_0,\frac{\theta}{k^n})=1$.

Consequently,

$F_S(\zeta_0,\zeta_0,\eta_0,\theta)=1$.

By part (2) of Definition 3 of the fuzzy S-metric, this implies that $\zeta_0=\eta_0$.

Theorem 3. Let $(\Omega,F_S,\triangle)$ be a fuzzy S-metric space and let $L:\Omega\to\Omega$ be a mapping. Assume that there exists a closed ball $B[\zeta_0,\varepsilon,t]\subset\Omega$ on which L satisfies a α-fuzzy contractive condition with contractive constant $\alpha \in (0,1)$. Moreover, suppose that

$$F_S(\zeta_0,\zeta_0,L(\zeta_0),\theta)>1-\varepsilon. \qquad (3)$$

Then the mapping L admits a unique fixed point in the closed ball $B[\zeta_0,\varepsilon,t]$.

The localization condition guarantees that the iterative sequence remains inside the closed ball, whereas the α-fuzzy contractive condition is employed solely to establish convergence and uniqueness.

Proof.

Let $\zeta_0 \in \Omega$ and define the sequence $\{\zeta_n\}$ by

$\zeta_1=L(\zeta_0), \quad \zeta_2=L(\zeta_1),\ldots,\zeta_{n+1}=L(\zeta_n), \quad n\geq 0.$

From the assumption

$F_S(\zeta_0,\zeta_0,L(\zeta_0),\theta)>1-\varepsilon,$

it follows that

$F_S(\zeta_0,\zeta_0,\zeta_1,\theta)=F_S(\zeta_0,\zeta_0,L(\zeta_0),\theta)>1-\varepsilon.$

Thus, $\zeta_1 \in B[\zeta_0,\varepsilon,t]$.

Assume inductively that $\zeta_1,\zeta_2,\ldots,\zeta_n \in B[\zeta_0,\varepsilon,t]$. Then

$F_S(\zeta_n,\zeta_n,L(\zeta_n),\theta)\&=F_S(\zeta_n,\zeta_n,\zeta_{n+1},\theta)$
$\&\geq F_S(\zeta_{n-1},\zeta_{n-1},\zeta_n,\theta)$
$\&>1-\varepsilon,$

which implies

$F_S(\zeta_n,\zeta_n,\zeta_{n+1},\theta)>1-\varepsilon.$

Hence, by induction, $\zeta_n \in B[\zeta_0,\varepsilon,t]$ for all n.

Moreover, using the property of the fuzzy S-metric, we have

$F_S(\zeta_0,\zeta_0,\zeta_n,\theta)\geq F_S(\zeta_0,\zeta_0,\zeta_1,\frac{\theta}{\alpha})\triangle F_S(\zeta_1,\zeta_1,\zeta_2,\frac{\theta}{\alpha})\triangle\ldots\triangle F_S(\zeta_{n-1},\zeta_{n-1},\zeta_n,\theta)>1-\varepsilon.$

Thus, $\{\zeta_n\}$ is contained in $B[\zeta_0,\varepsilon,t]$ and, there exists $\zeta_1 \triangle B[\zeta_0,\varepsilon,t]$ such that

$\zeta_n\rightarrow\zeta_1$ as $n\rightarrow\infty$ in $B[\zeta_0,\varepsilon,t]$.

Now, using the α-contractive property, we have

$F_S(L\zeta_n,L\zeta_n,L\zeta_1,\theta)\geq 1-\varepsilon\geq\frac{1}{\alpha}F_S(\zeta_n,\zeta_n,\zeta_1,\theta).$

Taking the limit as $n\rightarrow\infty$ gives

$\lim_{n\rightarrow\infty} F_S(L\zeta_n,L\zeta_n,L\zeta_1,\theta)=F_S(L\zeta_1,L\zeta_1,L\zeta_1,\theta)=1,$

so that

$\lim_{n\rightarrow\infty}\zeta_{n+1}=L(\zeta_1)=\zeta_1.$

Therefore, ζ_1 is a fixed point of L in $B[\zeta_0,\varepsilon,t]$.

For uniqueness, suppose $\zeta_2 \in B[\zeta_0,\varepsilon,t]$ is another fixed point of L. Then

$F_S(\zeta_2,\zeta_2,\zeta_1,\theta)=F_S(L(\zeta_2),L(\zeta_2),L(\zeta_1),\theta)\geq\frac{1}{\alpha}F_S(\zeta_2,\zeta_2,\zeta_1,\theta)\geq\frac{1}{\alpha^n}F_S(\zeta_2,\zeta_2,\zeta_1,\theta),$

for all $n \in N$. Since $\alpha\in(0,1)$, the only possibility is

$F_S(\zeta_2,\zeta_2,\zeta_1,\theta)=1$ for all $\theta>0$,

which implies $\zeta_2=\zeta_1$.

Hence, the fixed point is unique.

4. An Iterative Decision Framework's Fixed Point Analysis

In this section, we show how Theorem 3 can be applied to a basic computer science problem: the convergence of value iteration algorithms that arise in dynamic programming and reinforcement learning in the presence of uncertainty. Noisy rewards, imprecise transition probabilities, or insufficient system information are the natural causes of such uncertainty. The study of iterative decision-making processes under uncertainty has been extensively researched. While more modern approaches use fuzzy structures to simulate imprecision [7], classical results typically assume accurate metric spaces [28,29]. Theoretical connections to dynamical systems [30], which link fixed point results to convergence and stability in iterative processes, further strengthen this paradigm. Nevertheless, there is still a lack of research on the use of fuzzy S-metric spaces for value iteration, especially in localized contractive situations. By offering strict convergence guarantees in fuzzy settings where the metric encodes approximation and tolerance, our contribution closes this gap. Having constructed an fuzzy S-metric space of bounded value functions, we show that the usual value iteration operator satisfies the hypotheses of Theorem3. This extends classical convergence results to settings where uncertainty is measured systematically using a fuzzy metric by generating a unique fixed point corresponding to the optimal value function. Our theorem is especially useful in practical applications where iterations may only stabilize within subsets of the state space, due to its local character which allows for situations where global contractivity might fail to hold.

5. The Iterative Model's Analytical Setting

Consider a computational process where a deterministic rule is applied iteratively to update the state of a system. These models are found in learning-based update mechanisms, iterative optimization processes, and decision-making algorithms. The set of all admissible system states is denoted by Ω, and the update operator controlling the system's evolution is represented by

$L:\Omega\to\Omega$.

The state space Ω is equipped with a fuzzy S-metric F_S and a continuous t-norm $\triangle$, so that $(\Omega,F_S,\triangle)$ forms a fuzzy S-metric space in order to account for uncertainty, tolerance, or imprecision inherent in computational environments. The value $F_S(\zeta_0,\zeta_0,\eta_0,\theta)$ and another state ζ_0 at time parameter $\theta>0$.

Let S be the finite or countable set of system states and let

$\Omega=\{V:S\to R|V \text{ is bounded}\}$

denote the space of all bounded value functions.

We equip Ω with a fuzzy S-metric $F_S:\Omega^3\times(0,\infty)\to[0,1]$ defined by

$$F_S(V_1,V_1,V_2,\theta)=\exp\left(-\frac{1}{\theta}\sup_{s\in S}|V_1(s)-V_2(s)|\right),\theta>0.$$

Then $(\Omega,F_S,\triangle)$ forms a fuzzy S-metric space.

Consider the value iteration operator $L:\Omega\to\Omega$ defined by

$$(LV)(s)=\max_{\alpha\in A}\left[R(s,a)+\gamma_0\sum\nolimits_{s^{'}\in S}P(s^{'}|s,a)V(s^{'})\right],$$

where $R(s,a)$ denotes the reward function, $P(s^{'}|s,a)$ the transition probabilities, and $\gamma_0\in(0,1)$ the discount factor. The associated iterative algorithm is given by

$\zeta_{n+1}=L(\zeta_n),n\geq0$.

6. Contractive Nature of the Update Operator

We assume that the update operator L is contractive, within a small region of state space. More precisely, there are $\alpha\in(0,1)$, $\varepsilon\in(0,1)$, $\theta>0$, and a reference state $\zeta_0\in\Omega$ such that L satisfies an α-fuzzy contractive condition on the closed ball

$B[\zeta_0,\varepsilon,t]=\{\eta\in\Omega:F_S(\zeta_0,\zeta_0,\eta,\theta)\geq1-\epsilon\}$.

The condition takes into account that the updates are reducing the fuzzy separation between the data points between states, which is a useful property for convergence and stability in iterative methods.

Let $V_1,V_2\in\Omega$. By properties of the value iteration operator, we have

$$\sup_{s\in S}|LV_1(s)-LV_2(s)|\leq\gamma_0\sup_{s\in S}|V_1(s)-V_2(s)|.$$

Consequently,

$$\begin{aligned}F_S(LV_1,LV_1,LV_2,\theta)&=\exp\left(-\frac{1}{\theta}\sup_{s}|LV_1(s)-LV_2(s)|\right)\\&\geq\exp\left(-\frac{\gamma_0}{\theta}\sup_{s}|V_1(s)-V_2(s)|\right)\\&=(F_S(V_1,V_1,V_2,\theta))^{\gamma_0}.\end{aligned}$$

Hence,

L satisfies an α-fuzzy contractive condition on Ω with $\alpha=\gamma_0\in(0,1)$.

7. Local Invariance of the Iteration Domain

In addition to contractive-ness, we require that the initial update does not escape the prescribed local domain. Specifically, assume that

$$\alpha F_S(\zeta_0,\zeta_0,L(\zeta_0),\theta)>1-\varepsilon. \quad (4)$$

Inequality (4) ensures that the image of ζ_0 is stillclose enough to ζ_0 in a fuzzy manner. Therefore, the closed ball $B[\zeta_0,\varepsilon,t]$of L, to ensure that all iterates produced by the algorithm remain in a controlled neighborhood.

Let $\zeta_0\in \Omega$ be an initial approximation and fix $\varepsilon \in(0,1)$ and $t>0$.

Consider the closed ball

$B[\zeta_0,\varepsilon,t]=\{V\in\Omega:F_S(\zeta_0,\zeta_0,V,\theta)\geq1-\varepsilon\}$.

Assume that the initial approximation satisfies

$\alpha F_S(\zeta_0,\zeta_0,L(\zeta_0),\theta)>1-\varepsilon$.

This condition ensures that $L(\zeta_0)\in B[\zeta_0,\varepsilon,t]$ and that the value iteration remains locally confined within this closed ball.

8. Existence and Uniqueness of a Stable State

Under the above assumptions, all the hypotheses of Theorem 3 are satisfied. Therefore, the operator L admits a unique fixed point

$\eta_0^*\in B[\zeta_0,\varepsilon,t]$such that $L(\eta_0^*)=\eta_0^*$.

This fixed point corresponds to a point of stability of the iterative decision system. Uniqueness: It indicates that this result of this process does not depend upon which initial state selected within the admissible region, on the condition that the initial state lies in the closed ball $B[\zeta_0,\varepsilon,t]$.

Theorem 4. Let $(\Omega,F_S,\triangle)$ be the fuzzy S-metric space and let $L:\Omega\rightarrow\Omega$ be the value iteration operator.

If there exists a closed ball $B[\zeta_0,\varepsilon,t]\subset \Omega$ such that L satisfies an α-fuzzy contractive condition on $B[\zeta_0,\varepsilon,t]$ with $\alpha\in(0,1)$ and

$\alpha F_S(\zeta_0,\zeta_0,L(\zeta_0),\theta)>1-\varepsilon$,

then the value iteration algorithm admits a unique fixed point $V^*\in B[\zeta_0,\varepsilon,t]$,

that is,

$L(V^*)=V^*$.

9. Computational Interpretation

In computational terms, the fixed point attained represents a state of stability within the system, including but not limited to a decision fingerprint, an equilibrium point, a stationary point, an equilibrium solution, the equilibrium solution, a stationary solution, a solution results of an optimization routine or a converged state of an iterative learning process. By using the fuzzy S-metric formulation, this analysis is capable of dealing with uncertainties and approximations in computations that often happen in practice.

As regards the localness of the fixed point property, it has very important implications in the field of computer science applications since the globally contractive property might be too limiting there. It should be noted that convergence can be guaranteed once the sequence enters a stable region. This proposed framework is appropriate for implementation.

The fixed point V^* is the unique optimality function in the decision process. Theorem 4 ensures the uniqueness and stability of the converged solution of the value iteration algorithm in the environment of uncertainty captured by the fuzzy S-metric structure. Indeed, this theorem generalizes the traditional solution convergence in metric spaces into fuzzy computational environments.

10. Conclusion

In this work, the fixed point theorem in fuzzy S-metric spaces involving α-contractive mappings in a localized manner has been proven. This localization in the fixed point theorem allowed the theorem to be generalized in circumstances that that were not met in the globally fixed point theorem. Additionally, the localization in the fixed point theorem facilitated the determination of the fixed point of a function in circumstances that would have been difficult in the globally fixed point theorem. The applicability of the obtained theoretical outcomes was proved through a practical example taken from the value iteration process of the dynamic programming technique, where the fuzzy S-metric structure could successfully depict the uncertainty and imperfection of reward functions or transition process mechanisms. The results offered a significant analytical tool for investigating convergence concepts relative to computational approaches, optimization processes, and iterative algorithms in uncertain situations. Therefore, this research added value to the progress of fixed point theory within fuzzy S-metric spaces and indicated new pathways for more applications related to applied mathematics fields.

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Conflict of Interest Statement

The author confirms that there are no known financial or personal conflicts of interest that could have influenced the findings or interpretations presented in this study.

Generative AI Statement

The author declares that no Generative AI (GenAI) was used in the creation of this manuscript.

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